THE COLLEGES OF OXFORD UNIVERSITY

MATHEMATICS

SUNDAY 11 DECEMBER 2005

Time allowed: $2\frac{1}{2}$ hours

For candidates applying for Mathematics, Mathematics & Statistics, Computer Science, Mathematics & Computer Science, or Mathematics & Philosophy

Write your name, college (where you are sitting the test), and your proposed course (from the list above) in BLOCK CAPITALS

NAME:

COLLEGE:

COURSE:

Attempt all the questions. Each of the ten parts of Question 1 is worth 4 marks, and Questions 2,3,4,5 are worth 15 marks each, giving a total of 100.

Question 1 is a multiple choice question for which marks are given solely for the correct answers. Answer Question 1 on the grid on Page 2.

Write your answers to Questions 2,3,4,5 in the space provided, continuing on the back of this booklet if necessary. Note that in any of the Questions 2-5 you may use any earlier parts (even those you do not attempt) in your solutions to later parts.

THE USE OF CALCULATORS OR FORMULA SHEETS IS PROHIBITED.

1. For each part of the question on pages 3–7 you will be given four possible answers, just one of which is correct. Indicate for each part **A–J** which answer (a), (b), (c), or (d) you think is correct with a tick (✓) in the corresponding column in the table below. Please show any rough working in the space provided between the parts.

	(a)	(b)	(c)	(d)
A				
B				
C				
D				
E				
F				
G				
H				
I				
J				

A. The area of the region bounded by the curves $y = x^2$ and $y = x + 2$ equals

(a) $\frac{9}{2}$ (b) $\frac{7}{3}$ (c) $\frac{7}{2}$ (d) $\frac{11}{2}$

B. The equation

$$\left(x^2+1\right)^{10} = 2x - x^2 - 2$$

(a) has $x = 2$ as a solution;
(b) has no real solutions;
(c) has an odd number of real solutions;
(d) has twenty real solutions.

C. Given that

$$\log_{10} 2 = 0.3010 \quad \text{to 4 d.p. and that} \quad 10^{0.2} < 2$$

it is possible to deduce that

(a) 2^{100} begins in a 1 and is 30 digits long;
(b) 2^{100} begins in a 2 and is 30 digits long;
(c) 2^{100} begins in a 1 and is 31 digits long;
(d) 2^{100} begins in a 2 and is 31 digits long.

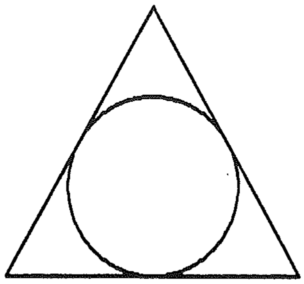
D. In the range $0 \leqslant x < 2\pi$ the equation

$$\cos(\sin x) = \frac{1}{2}$$

has

(a) no solutions;
(b) one solution;
(c) two solutions;
(d) three solutions.

E. A circle is inscribed in an equilateral triangle as shown in the diagram below.



The area of the circle, as a percentage of the triangle's, is approximately

(a) 40% (b) 50% (c) 60% (d) 70%

F. The fact that

$$6 \times 7 = 42,$$

is a counter-example to which of the following statements?

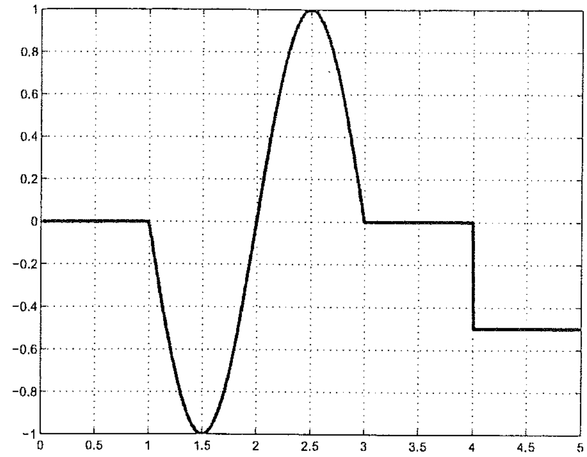
(a) the product of any two odd integers is odd;

(b) if the product of two integers is not a multiple of 4 then the integers are not consecutive;

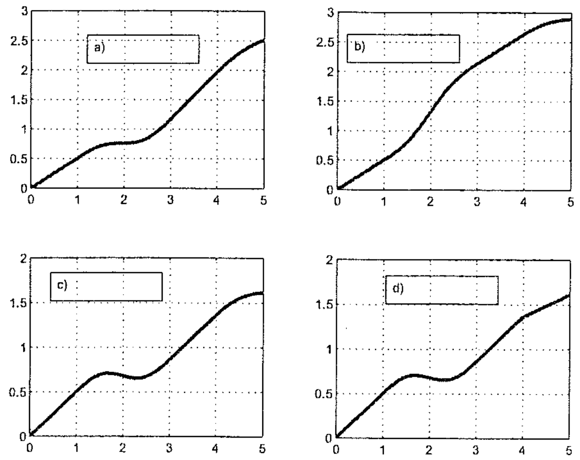
(c) if the product of two integers is a multiple of 4 then the integers are not consecutive;

(d) any even integer can be written as the product of two even integers.

G. A motorist drives along a curvy road. The acceleration $a(t)$ of her car as a function of time is graphed below.



One of the graphs below represents the distance $d(t)$ travelled by the motorist as a function of time. Which is the correct graph?



H. The four digit number 2652 is such that any two consecutive digits from it make a multiple of 13. Another number N has this same property, is 100 digits long, and begins in a 9. What is the last digit of N?

(a) 2 (b) 3 (c) 6 (d) 9

I. The curve with equation

$$x^{17} + x^3 + y^4 + y^{12} = 2$$

has

(a) neither the x-axis nor y-axis as a line of symmetry.
(b) the x-axis but not the y-axis as a line of symmetry;
(c) the y-axis but not the x-axis as a line of symmetry;
(d) both axes as lines of symmetry.

J. The numbers x and y satisfy

$$(x-1)^2 + y^2 \leqslant 1.$$

The largest that $x + y$ can be is

(a) 2 (b) $1+\sqrt{2}$ (c) 3 (d) $2+\sqrt{2}$

2. (i) Show, with working, that

$$x^3 - (1 + \cos\theta + \sin\theta)\,x^2 + (\cos\theta\sin\theta + \cos\theta + \sin\theta)\,x - \sin\theta\cos\theta, \qquad (1)$$

equals

$$(x-1)\left(x^2 - (\cos\theta + \sin\theta)\,x + \cos\theta\sin\theta\right)$$

Deduce that the cubic in (1) has roots

$$1, \qquad \cos\theta, \qquad \sin\theta.$$

(ii) Give the roots when $\theta = \frac{\pi}{3}$.

(iii) Find all values of θ in the range $0 \leqslant \theta < 2\pi$ such that two of the three roots are equal.

(iv) What is the greatest possible difference between two of the roots, and for what values of θ in the range $0 \leqslant \theta < 2\pi$ does this greatest difference occur?

Show that for each such θ the cubic (1) is the same.

3. (i) Find the co-ordinates of the turning points of

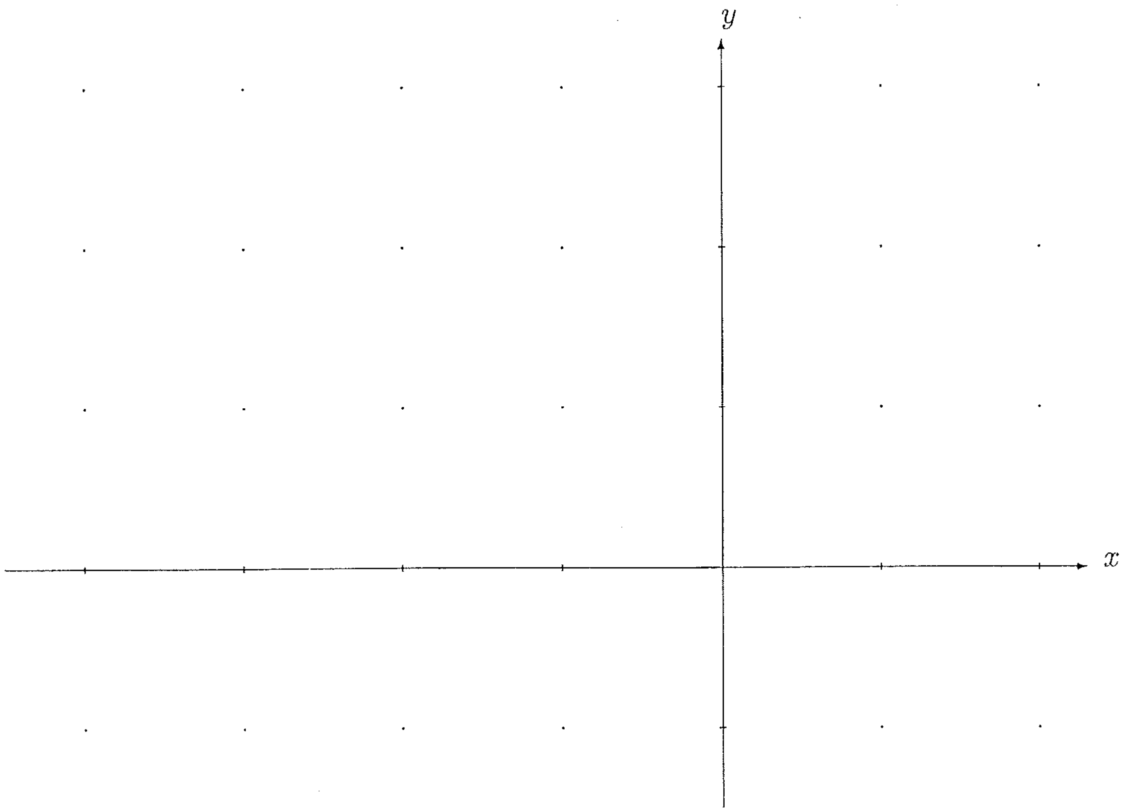
$$f(x) = e^x \left(2x^2 - x - 1\right).$$

(ii) Sketch the graph of $y = f(x)$ on the axes below for the range $-4 \leqslant x \leqslant 2$.

(iii) Now consider

$$g(x) = \begin{cases} e^x \left(2x^2 - x - 1\right) & \text{if } x < 1; \\ \sin(x - 1) & \text{if } x \geqslant 1. \end{cases}$$

Determine, with explanations, the maximum and minimum values of $g(x)$ as x varies over the real numbers.



4. An $n \times n$ square array contains 0s and 1s. Such a square is given below with $n = 3$.

0	0	1
1	0	0
1	1	0

Two types of operation C and R may be performed on such an array.

- The first operation C takes the first and second columns (on the left) and replaces them with a single column by comparing the two elements in each row as follows: if the two elements are the same then C replaces them with a 1, and if they differ C replaces them with a 0.
- The second operation R takes the first and second rows (from the top) and replaces them with a single row by comparing the two elements in each column as follows: if the two elements are the same then R replaces them with a 1, and if they differ R replaces them with a 0.

By way of example, the effects of performing R then C on the square above are given below.

0	0	1
1	0	0
1	1	0

$\xrightarrow{R}$

0	1	0
1	1	0

$\xrightarrow{C}$

0	0
1	0

(a) If R then C are performed (in that order) on a 2×2 array then only a single number (0 or 1) remains.

(i) Write down, in the grids on the next page, the eight 2×2 arrays which, when R then C are performed, produce a 1.

(ii) By grouping your answers accordingly, show that if

a	b
c	d

is amongst your answers to part (i) then so is

a	c
b	d

.

Explain why this means that doing R then C on a 2×2 array produces the same answer as doing C first then R.

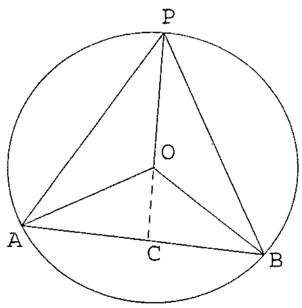
(b) Consider now an $n \times n$ square array containing 0s and 1s, and the effects of performing R then C or C then R on the square.

(i) Explain why the effect on the right $n - 2$ columns is the same whether the order is R then C or C then R. [This then also applies to the bottom $n - 2$ rows.]

(ii) Deduce that performing R then C on an $n \times n$ square produces the same result as performing C then R.

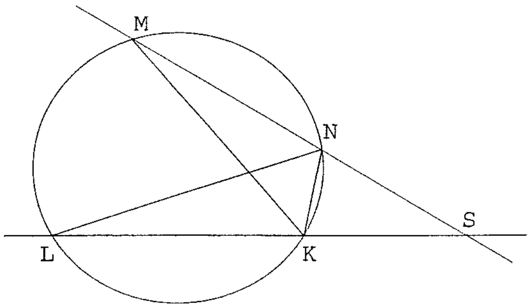
Turn Over

5. (a) Three points P, A, B lie on a circle which has centre O. The point C is where PO extends to meet AB as shown in the diagram below.



Show that $\measuredangle AOC = 2\measuredangle APC$ and $\measuredangle BOC = 2\measuredangle BPC$. Why does this mean that $\measuredangle APB$ is independent of the choice of the point P?

(b) Four points K, L, M, N lie on a circle and the lines LK and MN meet outside the circle at a point S, as shown in the diagram below.



Using part (a) and the Sine Rule show that

$$\frac{KS}{NS} = \frac{SM}{SL}.$$

[You may also assume that part (b) holds true in the special case when $M = N$ in which case the line SM is the tangent to the circle at M.]

(c) A tower has height h. Assuming the earth to be a perfect sphere of radius r, determine the greatest distance x from the top of the tower at which an observer can still see it.